

HOSSAM GHANEM

(1) Change Of Variables In Indefinite Integrals

$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$	$\int \sin x dx = -\cos x + C$
$\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$	$\int \cos x dx = \sin x + C$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\int \sec^2 x dx = \tan x + C$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$	$\int \csc^2 x dx = -\cot x + C$
$\tan^2 x = \sec^2 x - 1$	$\int \sec x \tan x dx = \sec x + C$
$\cot^2 x = \csc^2 x - 1$	$\int \csc x \cot x dx = -\csc x + C$

Example 1 Evaluate the following integral $\int (5x + 12)^{-4} dx$

Solution

$$I = \int (5x + 12)^{-4} dx$$

$$u = 5x + 12, \quad du = 5dx \quad \frac{1}{5} du = dx$$

$$I = \frac{1}{5} \int u^{-4} du = \frac{1}{5} \cdot \frac{-1}{3} u^{-3} + c = \frac{-1}{15} (5x + 12)^{-3} + c$$

Example 2 Evaluate $\int \sqrt[4]{3x + 5} dx$

Solution

$$u = 3x + 5$$

$$du = 3dx$$

$$\frac{1}{3} du = dx$$

$$I = \int \sqrt[4]{3x + 5} dx = \int (3x + 5)^{\frac{1}{4}} dx = \frac{1}{3} \int (u)^{\frac{1}{4}} du = \frac{1}{3} \cdot \frac{4}{5} (u)^{\frac{5}{4}} + c$$

$$= \frac{4}{15} (3x + 5)^{\frac{5}{4}} + c$$

Example 3 Find $\int \frac{(4x - 2)}{\sqrt{x^2 - x}} dx$

Solution

$$u = x^2 - x \quad du = (2x - 1) dx \quad 2du = (4x - 2) dx$$

$$I = 2 \int \frac{1}{\sqrt{u}} du = 2 \cdot 2\sqrt{u} + c = 4\sqrt{x^2 - x} + c$$

Example 4 Evaluate the following integral $\int \frac{1}{\sqrt{x}(\sqrt{x} + 1)^3} dx$

Solution

$$u = \sqrt{x} + 1 \quad du = \frac{1}{2\sqrt{x}} dx \quad 2du = \frac{1}{\sqrt{x}} dx$$

$$I = \int \frac{1}{\sqrt{x}(\sqrt{x} + 1)^3} dx = \int (\sqrt{x} + 1)^{-3} \cdot \frac{1}{\sqrt{x}} dx = 2 \int u^{-3} du$$

$$= 2 \cdot \frac{-1}{2} u^{-2} + c = -(\sqrt{x} + 1)^{-2} + c$$

Example 5 Find $\int \sqrt{s} \left(s^{\frac{3}{2}} - 7 \right)^5 ds$

Solution

$$u = s^{\frac{3}{2}} - 7 \quad du = \frac{3}{2} s^{\frac{1}{2}} ds \quad \frac{2}{3} du = \sqrt{s} ds$$

$$I = \int \sqrt{s} \left(s^{\frac{3}{2}} - 7 \right)^5 ds = \frac{2}{3} \int u^5 du = \frac{2}{3} \cdot \frac{1}{6} u^6 + c = \frac{1}{9} \left(s^{\frac{3}{2}} - 7 \right)^6 + c$$

Example 6 Find $\int \frac{1}{x^3} \left(1 + \frac{1}{x^2} \right)^{\frac{5}{3}} dx$

Solution

$$u = 1 + \frac{1}{x^2} \quad du = -2x^{-3} dx$$

$$u = 1 + x^{-2} \quad -\frac{1}{2} du = \frac{1}{x^3} dx$$

$$I = \int \frac{1}{x^3} \left(1 + \frac{1}{x^2} \right)^{\frac{5}{3}} dx = \int \left(1 + \frac{1}{x^2} \right)^{\frac{5}{3}} \cdot \frac{1}{x^3} dx = \frac{-1}{2} \int u^{\frac{5}{3}} du = \frac{-1}{2} \cdot \frac{3}{8} u^{\frac{8}{3}} + c = \frac{-3}{16} \left(1 + \frac{1}{x^2} \right)^{\frac{8}{3}} + c$$



Example 7 Evaluate the following integral $\int x(x^4 - 6x^2 + 9)^{17} dx$

Solution

$$x^4 - 6x^2 + 9 = (x^2 - 3)(x^2 - 3) = (x^2 - 3)^2$$

$$I = \int x(x^4 - 6x^2 + 9)^{17} dx = \int x[(x^2 - 3)^2]^{17} dx = \int (x^2 - 3)^{34} \cdot x dx$$

$$u = x^2 - 3 \quad du = 2x dx \quad \frac{1}{2} du = x dx$$

$$I = \frac{1}{2} \int u^{34} du = \frac{1}{2} \cdot \frac{1}{35} u^{35} + c = \frac{1}{70} (x^2 - 3)^{35} + c$$

Example 8 Find the antiderivative of $x\sqrt{x-3}$

Solution

$$u = x - 3 \quad du = dx \quad x = u + 3$$

$$I = \int x\sqrt{x-3} dx = \int (u+3)\sqrt{u} du$$

$$= \int (u+3)u^{\frac{1}{2}} du = \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du$$

$$= \frac{2}{5} u^{\frac{5}{2}} + 3 \cdot \frac{2}{3} u^{\frac{3}{2}} + c = \frac{2}{5} (x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c$$

Example 9 Evaluate the following integral $\int \frac{\sin \sqrt[3]{x}}{\sqrt[3]{x^2}} dx$

Solution

$$u = \sqrt[3]{x} \quad u = x^{\frac{1}{3}}$$

$$du = \frac{1}{3} x^{-\frac{2}{3}} dx \quad 3du = \frac{1}{\sqrt[3]{x^2}} dx$$

$$I = \int \frac{\sin \sqrt[3]{x}}{\sqrt[3]{x^2}} dx = \int \sin \sqrt[3]{x} \cdot \frac{1}{\sqrt[3]{x^2}} dx = 3 \int \sin u du = -3 \cos u + c = -3 \cos \sqrt[3]{x} + c$$

Example 10 Evaluate the following integral $\int x \sin x^2 \sqrt{1 + \cos x^2} dx$

Solution

$$u = 1 + \cos x^2 \quad du = -\sin x^2 \cdot 2x dx \quad \frac{-1}{2} du = x \sin x^2 dx$$

$$I = \int x \sin x^2 \sqrt{1 + \cos x^2} dx = \frac{-1}{2} \int \sqrt{u} du = \frac{-1}{2} \int u^{\frac{1}{2}} du = \frac{-1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + c = \frac{-1}{3} (1 + \cos x^2)^{\frac{3}{2}} + c$$

Example 11 Evaluate the following integral $\int (\sin 3x - \cos 3x)^2 dx$

Solution

$$(\sin 3x - \cos 3x)^2 = \sin^2 3x + \cos^2 3x - 2 \sin 3x \cos 3x = 1 - \sin 6x$$

$$I = \int (\sin 3x - \cos 3x)^2 dx = \int (1 - \sin 6x) dx = x + \frac{1}{6} \cos 6x + c$$

Homework

1 Evaluate $\int (2x - 3)^7 dx$

2 Evaluate $\int \sqrt{7x - 5} dx$

3 Evaluate $\int \sqrt[3]{2x - 1} dx$

4 Evaluate $\int \frac{7}{(2x + 9)^6} dx$

5 Evaluate $\int \frac{7}{(2x + 9)^6} dx$

6 Evaluate $\int \frac{1}{\sqrt{4x - 5}} dx$

7 Evaluate $\int \frac{dx}{\sqrt[5]{(8 - 3x)^6}}$

8 Evaluate $\int \frac{x^2 - 2}{(x^3 - 6x + 1)^6} dx$

9 Evaluate $\int x\sqrt{25 - x^2} dx$

10 Evaluate $\int \sqrt{s} \left(s^{\frac{3}{2}} + 1 \right)^5 ds$

11 Evaluate $\int \frac{1}{\sqrt{x}(\sqrt{x} - 5)^3} dx$

12 Evaluate $\int \left(1 + \frac{1}{x} \right)^{-3} \frac{1}{x^2} dx$

13 Evaluate $\int \left(1 + \frac{1}{x} \right)^{-7} \frac{1}{x^2} dx$

14 Find the antiderivative of $x\sqrt{x + 1}$

15 Find $\int x(x^4 + 2x^2 + 1)^{20} dx$

16 Evaluate $\int \frac{\sqrt{x^6 + 5x^4}}{x} dx, x > 0$

17 Evaluate $\int \cos(3x - 4) dx$

18 Evaluate $\int x \sec^2(2x^2 + 5) dx$

19 Evaluate $\int \frac{\cos \sqrt[3]{x}}{\sqrt[3]{x^2}} dx$

20 Evaluate $\int \frac{\sin 4x}{\sqrt{1 - \cos 4x}} dx$

21 Evaluate $\int (5 + \cos x)^3 \sin x dx$

22 Evaluate $\int \frac{\sin 3x}{(1 - \cos 3x)^5} dx$

23 Evaluate $\int \frac{\pi}{2} x \cos x^2 \sqrt{1 + \sin x^2} dx$

24 Evaluate $\int (\sin 2x + \cos 2x)^2 dx$

25 Show that $\int \sin^3 x dx = \frac{1}{3} \cos^3 x - \cos x + C$ where C is an arbitrary constant